International Journal of Advanced Information Science and Technology (IJAIST) ISSN: 2319:2682
Vol.3, No.10, October 2014 DOI:10.15693/ijaist/2014.v3i10.142-146

On Minimum Covering Of Star And Wrapped Butterfly Networks

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***Abstract*—A set S of vertices of a graph G = (V, E) is called a vertex cover, if each edge in E has at least one end point in S and the minimum cardinality taken over all vertex covering sets of G is called the vertex covering number denoted by β(G). This concept has also wide applications in industrial machine assignment, wireless sensor networks and in routing and fault tolerance algorithms. The star and wrapped butterfly networks were proposed as attractive alternatives to the popular hypercube for interconnecting processors on a parallel computer. In this paper, we present a characterization in connection with invertible networks. In particular, we examine their graph theoretic properties and present a polynomial time algorithm for solving VCP on star networks and for wrapped butterflies of even dimension.**

Index terms -Vertex cover, Edge cover, Invertible graphs, Star network, Wrapped Buttterfly.

I. Introduction

The problem of monitoring a network by placing an optimal set of nodes on which to strategically place controllers such that they can monitor the data going through every link in the network is modelled as the vertex covering problem (VCP) in graphs. The problem of finding a minimum vertex cover is a classical optimization problem in computer science and is a typical example of a NP hard optimization problem that has an approximation algorithm. Determining how well we can approximate vertex cover is one of the outstanding open problems in the complexity of approximation [1]. While vertex cover has a trivial 2-approximation algorithm, no better algorithms are known. The best approximation algorithm currently known for arbitrary graph is due to Karakostas [5]. On the other hand, minimum vertex cover is polynomial for several classes of graphs such as bipartite, chordal and graphs with bounded tree width etc. There has been many attempts to find the exact values of β(G) for special classes of graphs. Recently, Babak et.al.characterized minimum vertex cover in generalized Petersen graphs [3] and Madhavi.L et.al obtained exact values of minimum vertex cover and minimum edge cover in Mangoldt graphs [8]. Various graph characterizations have been done on covering relating to domination parameters and matching numbers and new graph parameters such as strong vertex cover, weak and balanced vertex covers and inverse vertex cover are being defined. Recently, statistical-mechanical methods have also been applied to study the vertex cover problem. Thus all these efforts carried out so far has resulted only in producing the approximate values for β(G). Although many approximation algorithms have been developed to solve the vertex cover problem no work has been done in finding the exact solution for the family of interconnection architectures. Among various kinds of popular networks, the star and wrapped butterfly networks have gained many researcher's efforts for its nice topological properties. They belong to the family of cayley graphs and are therefore vertex transitive [11]. These cayley graphs have optimal fault tolerance and logarithmic diameter. In this paper we focus on the wrapped butterfly and star interconnection networks for their optimal covering.

II. Preliminaries

Let G = (V, E) be a graph, where the set of points V called vertices represents a node and a collection E of unordered pairs of vertices called edges represent a link joining two nodes. A graph G is said to be a connected graph, if every pair of vertices of G has a path from one vertex to another. All the graphs considered here are finite, connected and undirected with no loops and multiple edges. We use the terms, graphs and networks, interchangeably. A matching in a graph is a set of pair wise non adjacent edges and a maximum matching is a matching that contains the largest possible number of edges denoted by μ(G). A perfect matching is a maximum matching which matches all vertices of the graph. It is also called a one factor. A matching is perfect if and only if it has $\frac{n}{2}$ edges where n is the number of vertices of G. The independence number α(G) of a graph G is the maximum number of non-adjacent vertices in G. A graph is said to be bridgeless if it contains no bridges. A graph G is bipartite if and only if it does not contain any odd cycles. A spanning tree T of a graph G is a tree that includes all of the vertices and some or all of the edges of G. Breadth first traversal of a graph also called breadth first search (bfs) of G corresponds to some kind of tree traversal goes a level at a time, left to right within a level, where a level is defined simply in terms of distance from the root of the tree.

III. Overview Of The Paper

The paper is organized as follows. In section III, we give the definitions of the vertex cover, edge cover and inverse cover problems and the importance of invertible graphs in electrical networks. We then present the topological properties of star interconnection networks. Section IV presents a characterization for invertible graphs and our important results on star networks, followed by section V which deals with wrapped butterfly networks. Section VI describes the MVC set algorithm for star interconnection networks and wrapped butterflies of even dimension. Finally, we conclude the paper by summarizing the main contributions and some future directions.

A.Minimum vertex cover problem

A set S of vertices of a graph G = (V, E) is called a vertex cover, if each edge in E has at least one end point in S. The minimum vertex cover is a vertex cover of smallest size. The cardinality of the minimum vertex cover set is called the covering number of G, denoted by β (G). Finding the minimum vertex cover is called the vertex cover problem.

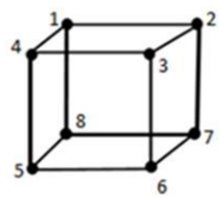

Figure 1. Minimum vertex cover set = {1, 3, 5, 7}

B. Minimum edge cover problem

A set C of edges of a graph G, such that each vertex of G is incident with at least one edge in C is called an edge cover. The minimum edge cover is an edge cover of smallest size and the cardinality of the minimum edge cover set is called the edge covering number denoted by β' (G). A perfect matching or a 1-factor is always a minimum edge cover. Finding the minimum edge cover is called the edge covering problem.

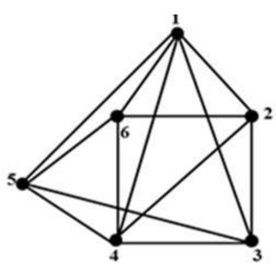

Figure 2 : Minimum edge cover set = {(1, 6), (2, 3), (4,5)}

C. Invertible graphs

Let D be a minimum vertex covering of G. A set S ⊆ V- D which is a vertex covering of G is called an inverse vertex covering of G with respect to the covering D. Then the inverse vertex covering number β^{-1}(G) is the order of smallest inverse vertex covering of G. We see that β^{-1}(G) need not exist for every graph. For example, the cycle graph on odd number of vertices has no inverse vertex cover. A graph G is said to be invertible if G admits an inverse vertex covering [3]. In figure 3a, the minimum vertex cover set D = {1,3,5,7} or {2,4,6,8} and β(G) = 4, whereas, the minimum inverse vertex cover set V - D = {2, 4, 6, 8} or {1, 3, 5, 7} with respect to D and therefore, β^{-1}(G) = 4. Since minimum inverse vertex cover exists the graph in figure 3a is invertible. Figure 3b is an example of a graph which is not invertible as it is 2-colorable.

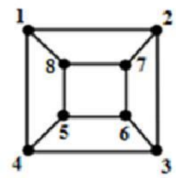

Figure: 3a

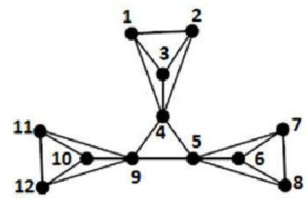

Figure: 3b

The covering and the inverse covering parameters play a vital role in coding theory, computer science, operations research, switching circuits, electrical networks, etc. [4]. In an electrical network, we have a set of primary nodes where the sensors are placed to monitor the entire system. In case, the system fails, we need to have another set of secondary nodes, to do the job in the complement. So if we want a vertex covering set in the complement set we look for the inverse covering set of G. Thus the covering and the inverse covering sets together facilitate the monitoring process. But this can happen only if the network is invertible. So we present a characterization for the invertible graphs We require the following theorems to prove our results.

Theorem 3.1: (Konig's) If G is a bipartite graph with no isolated vertices then α(G) = β'(G).

Theorem 3.2: [2] Let G be a bridgeless graph. Then G is invertible if and only if G is 2-colorable.

Theorem 3.3: [2]Let G be a bipartite graph with n number of vertices and $\bar{G}$ having no isolates. Then $\beta(\bar{G})$ = n – 2, where n is the number of vertices of G.

D.Topological properties of Star networks

The star graph was proposed as an attractive interconnection network for parallel processing, featuring smaller degree and diameter than a hypercube of comparable size. Due to its interesting properties, such as symmetry, sublogarithmic diameter, hamiltonicity and simple routing algorithms, the star graph has gained much attention and hence been widely studied in the past [6, 9, 10]. An n-dimensional star network also referred to as n-star or S_n, is an undirected

graph consisting of ! vertices and (−1) ! edges. Each vertex

is uniquely assigned a label $a_1, a_2, ..a_n$ which is the permutation of n distinct symbols {1, 2, 3, ..,n }. Two vertices are joined by an edge if the label of one vertex can be obtained from another by swapping the first element with any other element. For example, the vertex labelled 1234 will be connected with a vertex labelled 3214. Star graph of order 4 is shown below.

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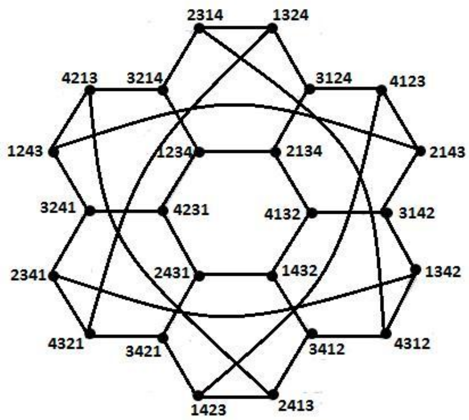

Figure 4 : Star network S_4

IV. OUR MAIN RESULTS

Theorem4.1: Let G be a graph with ( ) > 1. Then G is invertible if and only if G is bipartite.

Proof:

Assume that G is invertible. Then by theorem 3.5, G is bipartite. Conversely, assume G is bipartite. We shall prove that G is invertible. Since $\delta(G) > 1$, G will not have any pendent vertices. Therefore all vertices of G will lie on a cycle. Since G is bipartite, G will contain only even cycles. Each of these cycles are invertible and hence G is also invertible.

Theorem 4.2: Star networks Sn are invertible for all n.

Proof:

We observe that S_n is a bridgeless graph and is 2-colorable for all values of n. Therefore, from theorem 3.5, S_n is invertible.

Theorem 4.3: Let G be an n – dimensional star network.
Then β(G) = β'(G) = β^-1(G) = ! .

Proof:

Let G be an n – dimensional star network. Since G is bipartite, by Konig's theorem , we have α(G) = β (G). Now G has a 1-factor and so β'(G) — — = — .
This implies $\beta(G) = \beta^{-1}(G) = \beta'(G) = |_2| = !_2$.

Theorem 4.4: Let G be an n – dimensional star network. Then β( ) = —

Proof:

Let G be an n – dimensional star network. We know that G is invertible and hence bipartite for all n. Therefore by theorem 3.6, β($\bar{G}$) =| |– 2 = 6 ^{2} – 2.

V. WRAPPED BUTTERFLY NETWORK WB(r)

In a r-dimensional butterfly network [11], when the nodes of level 0 are merged with those in level r a new structure called the wrapped butterfly is obtained. The r-dimensional wrapped butterfly has $r.2^r$ nodes, each of degree 4 and $r.2^{r+1}$ edges. These networks are Hamiltonian for $r \geq 2$ [12]. In this section, we determine the exact value of the covering parameters of a wrapped butterfly network and prove that it is not invertible.

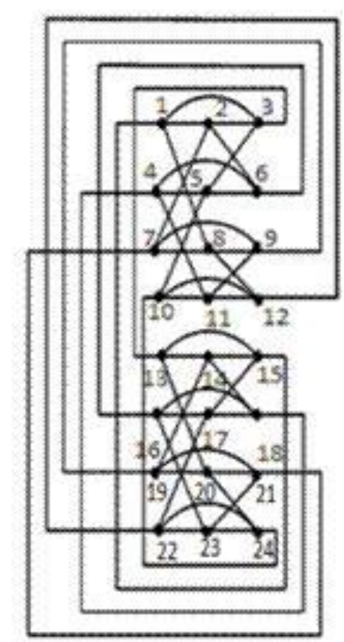

Figure 5: 3-dimensional wrapped butterfly network

Theorem 5.1:

Let G be an r-dimensional wrapped butterfly network. Then
(i) If r is even, then $\beta(G) = r.2^{r-1}$.
(ii) If r is odd, then β (G) = ⌊ + ⌋. .

Proof:

Let G be an r-dimensional wrapped butterfly network. If r is even , to cover the edges in the first row, we need
2 vertices. Therefore for 2 rows, we require exactly
2 . 2 vertices to cover all edges of G. That is, the minimum vertex cover number, $\beta(G) = r. 2^{-1}$.
If r is odd , to cover the edges in the first row, we need ⌊ $^{+1}$2⌋ vertices.

Therefore for 2 rows, we require exactly $\lfloor \frac{+1}{2} \rfloor . 2$ vertices to cover all edges of G. That is, the minimum vertex cover number, $\beta(G) = \lfloor \frac{+1}{2} \rfloor . 2$.

Theorem 5.2:
Let G be an r-dimensional wrapped butterfly network. Then for all r, $\beta'(G) = r.2^{r-1}$.

Proof:
Let G be an r-dimensional wrapped butterfly network.
If r is even, for minimum edge cover, we choose $\frac{}{2} . 2$ edges to cover all the vertices of G. This implies that $\beta'(G) = r.2^{r-1}$.
If r is odd, we need $\lfloor \frac{}{2} \rfloor . 2 + 2^{r-1}$ edges to cover all vertices of G. That is the minimum edge covering number, $\beta'(G) = \lfloor \frac{}{2} \rfloor . 2 + 2^{r-1}$.

$\Rightarrow \beta'(G) = (\frac{}{2} - \frac{1}{2}) . 2^r + 2^{r-1}$.

$\Rightarrow \beta'(G) = (r - 1). 2^{r-1} + 2^{r-1}$.

$\Rightarrow \beta'(G) = r. 2^{r-1}$.

Theorem 5.3: Wrapped butterfly networks are invertible for even values of r.

Proof:
Let G be an r-dimensional wrapped butterfly network. Since WBF(r) does not contain odd cycles for even values of r, by theorem 4.1, WBF(r) is invertible for even dimension.

V. An Algorithm For Finding A Minimum Vertex Cover Set In Star Networks And Wrapped Butterflies Of Even r.

In this section we introduce an algorithm to find the minimum vertex cover (MVC) set of star network and wrapped butterfly network with even dimension G. In this algorithm we use some variables which are defined below: h is the height of the bfs tree of G, $L_i(x)$ is the set of all vertices of G in level L_i of the BFS tree.

Input: A star network or wrapped butterfly network with even dimension G

Output: A MVC set S

1. Construct the bfs tree for G
2. $i \leftarrow 1$; $S \leftarrow$ empty set;
3. While $(i \leq h)$
4. $S \leftarrow S \cup L_i(x)$;
5. $i \leftarrow i + 2$;
6. endwhile
7. Return S

There are efficient parallel algorithms in the literature to construct a bfs tree of a graph G. We can use any of them (for one example, see [7]) for the first step of the algorithm. Time complexity of the bfs algorithm is O(V + E) and the time for "while" loop that is executed is O(h), and hence the total running time is O(V + E + h).

The following theorem gives the proof of correctness.

Theorem 6.1: Let T be a bfs tree rooted at v. Then the vertices in the alternate levels of T starting from level L_i forms a MVC set of G.

Proof:
Let T be a bfs tree BFS(v), then all vertices incident to v, will be in level L_2 of T. We choose the vertex v. Since the vertex v covers all the edges incident to both vertices in L_2 and v, we skip level L_2 and choose all vertices in L_3. Similarly those vertices in L_3 will cover all the edges incident to both vertices in L_3 and L_4 and hence we skip L_4. Repeating this process until we reach L_i where i = h + 1 we get the MVC set S = $\bigcup^{h+1}_{=0} \, _{+2}$. Since G is bipartite, there should not be any vertices adjacent to vertices of the same level. Therefore, S covers all the edges of G. Hence S is a vertex covering set of G. We claim that S is minimum. Suppose if S is not minimum, then there exists a covering D such that $|D| < |S|$. If such a set exists then S is not a covering set, which is a contradiction. Therefore, S is the MVC set of G. Hence the proof.

VI. Conclusion

`In this paper we have characterized invertible graphs and using that we have shown that the star networks and wrapped butterflies of even dimension are invertible. Further a MVC set algorithm is developed which runs in polynomial time and the exact values of the covering parameters of star networks and its complement are also obtained. The above results can also be applied to study the covering parameters of various other parallel architectures like pancake and pyramid networks.

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