Alpha – Weakly Continuous Mappings In Topological Spaces

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***Abstract*—In the literature various kinds of mappings between topological spaces have been defined. We introduce the notion of a new class of mappings called alpha-weakly continuous mappings and investigate its several properties and characterizations. Its connection with other existing concepts such as alpha-continuous and weakly alpha-continuous mappings are also investigated.**
Simulation results shows that the mobility oriented trust system provides better detection efficiency, good malicious node detection, low false positive and delay constraint.

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Key Words and Phrases: Topological Space, $\alpha-open$ set, $\alpha-closed$ set, $\alpha-weakly$ continuous, $\alpha-connected$ space

I. Introduction

The notion of $\alpha-open$ set (originally called $\alpha-sets$) in topological space was introduced by Njastad$[1965]$. Since then, it has been widely investigated in the literature. Throughout this paper, $(X,\tau)\,(simply\ X)$ always mean topological space. A subset A of (X,τ) is called $\alpha-open$ $[Njastad;\ 1965]$ if $A\subseteq Int[Cl(Int(A))]$. The complement of an $\alpha-open$ set is called an $\alpha-closed$ set. The intersection of all $\alpha-closed$ sets containing A is called the $\alpha-closure$ of A, denoted by $Cl_{\alpha}(A)$. A subset A is $\alpha-closed$ if and only if $A=Cl_{\alpha}(A)$. A point $x\in X$ is said to be an $\alpha-\text{int}erior$ point of A if there exists an $\alpha-open$ set U containing x such that $U\subseteq A$. The set of all $\alpha-\text{int}erior$ points of A is said to be $\alpha-\text{int}erior$ of $A\,[Mashhour\ \&\ El\text{-}Deeb;\ 1983]$ and denoted by $Int_{\alpha}(A)$. We denote the family of all $\alpha-open$ sets of (X,τ) by τ^{α}. It is shown in $[Njastad;\ 1965]$ $(see\ also\ Ohba\ \&\ Umehara;\ 2000)$ that each of $\tau\subseteq\tau^{\alpha}$ and τ^{α} is a topology on X.

II. ALPHA – WEAKLY CONTINUOUS MAPPINGS

Definition 1. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $\alpha-weakly$ continuous if for each $x\in X$ and each open sets V containing $f(x)$, there exists an $\alpha-open$ set U containing x such that $f(U)\subseteq Cl_{\alpha}(V)$.

Definition 2. $[Mashhour\ \&\ El\text{-}Deeb;\ 1983]$. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $\alpha-continuous$ if $f^{-1}(V)\in\tau^{\alpha}$ for every $V\in\sigma$, and, equivalently, if for each $x\in X$ and each open sets V of Y containing $f(x)$, there exists $U\in\tau^{\alpha}$ with $x\in U$ such that $f(U)\subseteq V$.

We remark that every $\alpha-continuous$ mapping is $\alpha-weakly$ continuous, but the converse is not true as the following example shows.

Example 3. Let X and Y be both the set of real numbers. Let τ be the usual topology for X and σ the cocountable topology for Y. Then the identity mapping $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous and not $\alpha-continuous$. Firstly we show that f is not $\alpha-continuous$. Let $\sqrt{2}\in I=$ the set of irrational numbers. Then $I\in\sigma$ as $R-I$ is countable. Let $U\subseteq I$ such that $\sqrt{2}\in U$. Then $Int\left[Cl\left(Int(U)\right)\right]=\phi$. Hence $U\notin\tau^{\alpha}$. Thus f is not $\alpha-continuous$.

Now we show that f is $\alpha-weakly$ continuous. Let $x\in V$ and $V\in\sigma$. Then $R-V$ is countable. We show that

$Cl_\alpha(V)=Y.$ Let $y\in W\in\sigma^\alpha.$ If W is countable, then $Int(W)=\phi.$ Hence $Int\left[Cl(Int(W))\right]=\phi.$ This shows that $W\notin\sigma^\alpha.$ Hence W is uncountable. Since $Y-V$ is countable. Hence $W\cap V\neq\phi.$ It shows that $Cl_\alpha(V)=Y.$ Now by definition it follows that f is $\alpha-weakly$ continuous.

Definition 4. $[Noiri;\ 1987]$. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be weakly $\alpha-continuous$ continuous if for each $x\in X$ and each open set V of Y containing $f(x)$, there exists $U\in\tau^\alpha$ containing x such that $f(U)\subseteq Cl(V).$

Theorem 5. Every $\alpha-weakly$ continuous function is weakly $\alpha-continuous$.

Proof. We know that $\sigma\subseteq\sigma^\alpha.$ Hence $Cl_\alpha(V)\subseteq Cl(V).$ The theorem follows.

Theorem 6. A mapping $f:(X,\tau)\to(Y,\sigma)$ is $\alpha-weakly$ continuous if and only if for every open set V in Y, $f^{-1}(V)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right].$

Proof. Let $x\in X$ and V an open set containing $f(x)$. Then $x\in f^{-1}(V)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right].$ Put $U=Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right].$ Then U is $\alpha-open$ and $f(U)\subseteq Cl_\alpha(V).$ Conversely, let V be an open set of Y and $x\in f^{-1}(V).$ Then there exists an open set U in X such that $x\in U$ and $f(U)\subseteq Cl_\alpha(V).$ Therefore, we have $x\in U\subseteq f^{-1}\left[(Cl_\alpha(V))\right]$ and hence $x\in Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right].$ This proves that $f^{-1}(V)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right].$

Theorem 7. Let $f:(X,\tau)\to(Y,\sigma)$ be $\alpha-weakly$ continuous. If V is a clopen (both closed and open) subset of Y such that $f(x)\in V$, then $f^{-1}(V)$ is clopen in (X,τ^α).

Proof. Let $x\in X$ and V be a clopen subset of (Y,σ) such that $f(x)\in V$. Then there exists $U\in\tau^\alpha$ containing x such that $f(U)\subseteq Cl_\alpha(V).$ Since $\sigma\subseteq\sigma^\alpha$ implies every clopen subset of (Y,σ) is also a clopen subset of (Y,σ^α). Hence $x\in U$ and $f(U)\subseteq V$ and so $x\in U\subseteq f^{-1}(V).$ This shows that $f^{-1}(V)\in\tau^\alpha.$ Since $Y-V$ is a clopen in (Y,σ), so $f^{-1}(Y-V)\in\tau^\alpha.$ But $f^{-1}(Y-V)=X-f^{-1}(V).$ Therefore $f^{-1}(V)$ is closed in $(X,\tau^\alpha).$ Hence $f^{-1}(V)$ is an $\alpha-clopen$ (both $\alpha-closed$ and $\alpha-open$) set in $(X,\tau^\alpha).$

Theorem 8. The following are equivalent for a mapping $f:(X,\tau)\to(Y,\sigma)$.

(1) f is $\alpha-weakly$ continuous.

(2) $f^{-1}(V)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V))\right]$ for every open subset V of $(Y,\sigma).$

(3) $Cl_\alpha\left[f^{-1}(Int_\alpha(V))\right]\subseteq f^{-1}(V)$ for every $closed$ subset V of $(Y,\sigma).$

Proof. $(1)\Leftrightarrow(2):$ Follows from Theorem 6.

$(2)\Rightarrow(3):$ Let V be a closed subset of $(Y,\sigma).$ Then $Y-V$ is an open set in $(Y,\sigma).$ So by hypothesis

$$f^{-1}(Y-V)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(Y-V))\right]$$
$$=Int_\alpha\left[f^{-1}(Y-Int_\alpha(V))\right]$$
$$=X-Cl_\alpha\left[f^{-1}(Int_\alpha(V))\right].$$

Thus $Cl_\alpha\left[f^{-1}(Int_\alpha(V))\right]\subseteq f^{-1}(V).$

$(3)\Rightarrow(1):$ Let $x\in X$ and let $f(x)\in V\in\sigma.$ So $Y-V$ is a $closed$ set in $(Y,\sigma).$ So by hypothesis $Cl_\alpha\left[f^{-1}(Int_\alpha(Y-V))\right]\subseteq f^{-1}(Y-V)$. Thus $x\notin Cl_\alpha\left[f^{-1}(Int_\alpha(Y-V))\right].$ Hence there exists $U\in\tau^\alpha$ such that $x\in U$ and $U\cap f^{-1}(Int_\alpha(Y-V))=\phi$ which implies that $f(U)\cap Int_\alpha(Y-V)=\phi.\Rightarrow f(U)\subseteq Y-Int_\alpha(Y-V)\Rightarrow f(U)\subseteq Cl_\alpha(V).$ This shows that f is $\alpha-weakly$ continuous..

Theorem 9. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\mu)$ be any mappings and let $gof:(X,\tau)\to(Z,\mu)$ be the composition.

(1). If $g:(X,\tau^{\alpha})\to(Y,\sigma^{\alpha})$ is an open surjection and gof is $\alpha-weakly$ continuous, then g is $\alpha-weakly$ continuous.

(2). If $g:(X,\tau^{\alpha})\to(Y,\sigma^{\alpha})$ is continuous and g is $\alpha-weakly$ continuous, then gof is $\alpha-weakly$ continuous.

Proof. (1) Let $y\in Y$. Since f is a surjection, there exists $x\in X$ such that $f(x)=y$. Let $V\in\mu$ contain $(gof)(x)$. Since gof is $\alpha-weakly$ continuous, there exists, $U\in\tau^{\alpha}$ containing x such that $(gof)(U)\subseteq Cl_{\alpha}(V)$. By hypothesis, it follows that $W=f(U)\in\sigma^{\alpha}$ and contains $f(x)=y$. Thus $g(W)\subseteq Cl_{\alpha}(V)$. Hence g is $\alpha-weakly$ continuous.

(2) Let $x\in X$ and $W\in\mu$ such that $(g\circ f)(x)=g(f(x))\in W$. Let $y=f(x)$. Since g is $\alpha-weakly$ continuous. So there exists $V\in\sigma^{\alpha}$ such that $g(y)\in V$ and $g(V)\subseteq Cl_{\alpha}(W)$. Let $U=g^{-1}(V)$. Then U is open in (X,τ^{α}) as f is $\alpha-continuous$. Now $(g\circ f)(U)=g(f(f^{-1}(V)))\subseteq g(V)$. Then $x\in U\in\tau^{\alpha}$ and $(g\circ f)(U)\subseteq Cl_{\alpha}(W)$. Hence $g\circ f$ is $\alpha-weakly$ continuous

Theorem 10. Let $f:X\to Y$ be a mapping and $g:X\to X\times Y$ be the graph mapping of f, given by $g(x)=(x,f(x))$ for every point $x\in X$. Then f is $\alpha-weakly$ continuous if and only if g is $\alpha-weakly$ continuous.

Proof. **Necessity**. Suppose that f is $\alpha-weakly$ continuous. Let $x\in X$ and $g(x)\in W\in\tau\times\sigma$. There exist $U_1\in\tau$ and $V\in\sigma$ such that $(x,f(x))\in U_1\times V\subseteq W$. Since f is $\alpha-weakly$ continuous, there exists $U_2\in\tau^{\alpha}$ containing x such that $f(U_2)\subseteq Cl_{\alpha}(V)$. Put $U=U_1\cap U_2$, then we have $x\in U\in\tau^{\alpha}$ and $g(U)\subseteq Cl_{\alpha}(W)$. This indicates that g is $\alpha-weakly$ continuous.

Sufficiency. Suppose that g is $\alpha-weakly$ continuous. Let $x\in X$ and V be any open set containing $f(x)$. Then $X\times V$ is an open set in $X\times Y$ containing $g(x)$. Since g is $\alpha-weakly$ continuous, there exists an $\alpha-open$ set U in X containing x such that $g(U)\subseteq Cl_{\alpha}(X\times V)$. It follows from Lemma 4 of $[Noiri;1978]$, that $Cl_{\alpha}(X\times V)\subseteq X\times Cl_{\alpha}(V)$. Since g is the graph mapping of f, we have $f(U)\subseteq Cl_{\alpha}(V)$. This shows that f is $\alpha-weakly$ continuous.

Let $\{(x_{\lambda},\tau_{\lambda}):\lambda\in\Lambda\}$ and $\{(y_{\lambda},\sigma_{\lambda}):\lambda\in\Lambda\}$ be any two families of spaces with the same index set Λ. Let $f_{\lambda}:(X_{\lambda},\tau_{\lambda})\to(Y_{\lambda},\sigma_{\lambda})$ be a function for each $\lambda\in\Lambda$. Let $f:(\prod X_{\lambda},\prod\tau_{\lambda})\to(\prod Y_{\lambda},\prod\sigma_{\lambda})$ denote the product function defined by $f(x_{\lambda}:\lambda\in\Lambda)=(f(x_{\lambda}):\lambda\in\Lambda)$ for every $(x_{\lambda}:\lambda\in\Lambda)\in\prod X_{\lambda}$. Moreover, let $p_{\mu}:\prod_{\lambda\in\Lambda}X_{\lambda}\to X_{\mu}$ and $q_{\mu}:\prod_{\lambda\in\Lambda}Y_{\lambda}\to Y_{\mu}$ be the natural projections. Then, we have the following result.

Theorem 11. The product function $f:(\prod X_{\lambda},\prod\tau_{\lambda})\to(\prod Y_{\lambda},\prod\sigma_{\lambda})$ is $\alpha-weakly$ continuous if and only if $f_{\lambda}:(X_{\lambda},\tau_{\lambda})\to(Y_{\lambda},\sigma_{\lambda})$ is $\alpha-weakly$ continuous for each $\lambda\in\Lambda$.

Proof. **Necessity**. Suppose that that f is $\alpha-weakly$ continuous. Let μ be an arbitrary fixed index of Λ. Since q_{μ} is continuous, by Theorem 9 $q_{\mu}\circ f=f_{\mu}\circ p_{\mu}$ is $\alpha-weakly$ continuous. Moreover, p_{μ} is an open continuous surjection and hence by Theorem 9 f_{μ} is $\alpha-weakly$ continuous.

Sufficiency. Suppose that f_{λ} is $\alpha-weakly$ continuous for each $\lambda\in\Lambda$. Let $x=(x_{\lambda}:\lambda\in\Lambda)\in\prod X_{\lambda}$ and $f(x)\in W\in\prod\sigma_{\lambda}$. There exists a basic open set $\prod V_{\lambda}$

such that $f(x) \in \prod V_\lambda \subseteq W$ and $\prod V_\lambda = \prod_{i=1}^{n} V_{\lambda_i} \times \prod_{\lambda \neq \lambda_i}^{n} Y_{\lambda_i}$, where $V_\lambda \in \sigma_\lambda$ for each $\lambda = \lambda_1, \lambda_2, ..., \lambda_n$. Since f_λ is $\alpha - weakly$ continuous, for each $\lambda = \lambda_1, \lambda_2, ..., \lambda_n$ there exists $U_\lambda \in \tau_\lambda^\alpha$ containing x_λ such that $f(U_\lambda) \subseteq Cl_\alpha(V_\lambda)$. Now, let us put $U = \prod_{i=1}^{n} U_{\lambda_i} \times \prod_{\lambda \neq \lambda_i}^{n} X_{\lambda_i}$, then we have $x \in U \in (\prod \tau_\lambda)^\alpha$ and $f(U) \subseteq Cl_\alpha(W)$. This indicates that f is $\alpha - weakly$ continuous.

Definition 12. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be weakly continuous if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(U) \subseteq Cl(V)$.

Every weakly continuous function is $\alpha - weakly$ continuous but the converse is not true by the following example.

Example 13. Let $X = Y = \{a,b,c\}, \tau = \{\phi, \{c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a,b\}, Y\}$. Let $f:(X,\tau) \to (Y,\sigma)$ be the identity function. Then f is $\alpha - weakly$ continuous but it is not weakly continuous.

Lemma 14. A function $f:(X,\tau) \to (Y,\sigma)$ is $\alpha - weakly$ continuous if and only if $f:(X,\tau^\alpha) \to (Y,\sigma)$ is weakly continuous.

Theorem 15. A function $f:(X,\tau) \to (\prod Y_\lambda, \prod \sigma_\lambda)$ is $\alpha - weakly$ continuous if and only if $q_\lambda \circ f:(X,\tau) \to (Y_\lambda, \sigma_\lambda)$ is $\alpha - weakly$ continuous for each $\lambda \in \Lambda$.

Proof. This follows immediately from Lemma 14 and the fact that a function $f:(X,\tau) \to (\prod Y_\lambda, \prod \sigma_\lambda)$ is weakly continuous if and only if $q_\lambda \circ f:(X,\tau) \to (Y_\lambda, \sigma_\lambda)$ is continuous for each $\lambda \in \Lambda$.

Theorem 16. If $f:(X,\tau) \to (Y,\sigma)$ is $\alpha - weakly$ continuous and (Y,σ) is regular, then f is continuous.

Proof. Let x be any point of X and V any open set of (Y,σ) containing $f(x)$. Since (Y,σ) is regular, there exists $W \in \sigma$ such that $f(x) \in W \subseteq Cl(W) \subseteq V$. Since f is $\alpha - weakly$ continuous, there exists $U \in \tau^\alpha$ containing x such that $f(U) \subseteq Cl_\alpha(W) \subseteq Cl(W) \subseteq V$. Therefore, f is $\alpha - continuous$ by Theorem 1 in [Mashhour et al; 1983] and hence it is continuou by the remark of [Mashhour et al; 1983].

Theorem 17. If $f: X \to Y$ is an $\alpha - weakly$ continuous mapping and Y is Hausdorff, then the graph $G(f)$ is an $\alpha - closed$ set of $X \times Y$.

Proof. Let $(x,y) \notin G(f)$. Then, we have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets W and V such that $f(x) \in W$ and $y \in V$. Since f is $\alpha - weakly$ continuous, there exists an $\alpha - open$ set U containing x such that $f(U) \subseteq Cl_\alpha(W)$. Since W and V are disjoint, we have $V \cap Cl_\alpha(W) = \phi$ and hence $V \cap f(U) = \phi$. This shows that $(U \times V) \cap G(f) = \phi$. It follows that $G(f)$ is $\alpha - closed$.

Definition 18. By an $\alpha - weakly$ continuous retraction, we mean an $\alpha - weakly$ continuous mapping $f: X \to A$, where $A \subseteq X$ and $f_A = f/A$ is the identity mapping on A.

Theorem 19. Let $A \subseteq X$ and $f: X \to Y$ be an $\alpha - weakly$ continuous retraction of X onto A. If X is a Hausdorff space, then A is an $\alpha - closed$ set in X.

Proof. Suppose that A is not an $\alpha - closed$ set in X. Then there exists a point $x \in Cl_\alpha(A) - A$. Since f is $\alpha - weakly$ continuous retraction, we have $f(x) \neq x$. Since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Thus we get $U \cap Cl_\alpha(V) = \phi$. Now, let W be any $\alpha - open$ set in X containing x. Then $U \cap W$ is an $\alpha - open$ set containing x and hence $(U \cap W) \cap A \neq \phi$ because $x \in Cl_\alpha(A)$. Let $y \in (U \cap W) \cap A$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \notin Cl_\alpha(V)$. This gives that $f(W) \not\subset Cl_\alpha(V)$. This contradicts that f is $\alpha - weakly$ continuous. Hence A is $\alpha - closed$ in X.

Theorem 20. Let $f:(X,\tau)\to(Y,\sigma)$ be $\alpha-weakly$ continuous and A a subset of X such that if either $A\in PO(X,\tau)$ or $A\in SO(X,\tau)$. Then the restriction $f_A:(A,\tau_A)\to(Y,\sigma)$ is $\alpha-weakly$ continuous.

Proof. Since either $A\in PO(X,\tau)$ or $A\in SO(X,\tau)$. It follows from lemma 1.1 of $[Mashhour\ et\ al;1983]$ and $[\text{Reilly \& vamanamurthy;1984}]$ that $\tau^\alpha/A\subseteq(\tau/A)^\alpha$. Since f is $\alpha-weakly$ continuous, for each $x\in A$ and each $V\in\sigma$ containing $f(x)$, there exists $U\in\tau^\alpha$ containing x such that $f(U)\subseteq Cl_\alpha(V)$. Put $U_A=U\cap A,$ then we have $x\in U_A\in(\tau/A)^\alpha$ and $(f/A)(U_A)\subseteq Cl_\alpha(V).$ This indicates that f/A is $\alpha-weakly$ continuous.

III. ALPHA – CONNECTED SPACES

Definition 21. A space X is said to be $\alpha-connected$ if X can not be written as the disjoint union of two non-empty $\alpha-open$ sets.

Every $\alpha-connected$ space is connected but the converse may not be true.

It is shown in Theorem 4 of $[Long\ \text{et al};1973]$ $(resp.\ Theorem\ 3\ of\ [Noiri;1974])$ that connectedness is invariant under almost continuous (resp. weakly continuous) surjections. It is also known that $S-cnnectedness$ is invariant under semi-continuous surjections. In $[Noiri\ \&\ Ahmad;1985]$ it is proved that the semi-weakly continuous image of an $S-connected$ space is connected. In $[Latif;1995]$ we prove that $S-connectedness$ is invariant semi-weakly semi-continuous mapping. However, we have the following.

Theorem 22. If X is an $\alpha-connected$ space and $f:X\to Y$ is $\alpha-weakly$ connected surjection, then Y is connected.

Proof. Suppose that Y is not connected. Then there exist two non-empty open sets V_1 and V_2 of Y such that $V_1\cap V_2=\phi$ and $V_1\cup V_2=Y.$ Hence, we have $f^{-1}(V_1)\cap f^{-1}(V_2)=\phi$, $f^{-1}(V_1)\cup f^{-1}(V_2)=X$ and $f^{-1}(V_1)\neq\phi\neq f^{-1}(V_2)$ because f is surjection. By Theorem 8, we have $f^{-1}(V_i)\subseteq Int_\alpha\left[f^{-1}(Cl_\alpha(V_i))\right],$ for $i=1,2.$ Since each V_i is clopen and hence also $\alpha-clopen.$ We obtain $f^{-1}(V_i)\subseteq Int_\alpha\left[f^{-1}(V_i)\right]$ and hence $f^{-1}(V_i)$ is $\alpha-open$ for $i=1,2.$ This implies that X is not $\alpha-connected$. Therefore Y is connected.

Theorem 23. If X is an $\alpha-connected$ space and $f:X\to Y$ is $\alpha-continuous$ mapping with the closed graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two distinct points x_1,x_2 in X such that $f(x_1)\neq f(x_2)$. Since the graph $G(f)$ is closed and $(x_1,f(x_2))\notin G(f),$ there exist open sets U and V containing x_1 and $f(x_2)$, respectively, such that $f(U)\cap V=\phi.$ Since f is $\alpha-continuous$, U and $f^{-1}(V)$ are non-empty disjoint $\alpha-open$ sets. It follows that X is not $\alpha-connected$. Therefore f is constant.

Corollary 24. $[Thompson;1981].$

Let X be $\alpha-connected$. If $f:X\to Y$ is a continuous mapping with the closed graph, then f is constant.

Proof. Since every continuous mapping is $\alpha-continuous,$ this is an immediate consequence of Theorem 23.

Definition 25. $[Maheshwari\ \&\ Thakur;1980].$ A function $f:(X,\tau)\to(Y,\sigma)$ is an $\alpha-irresolue$ mapping if and only if the inverse image of every $\alpha-open$ set in Y is an $\alpha-open$ set in $X.$

Theorem 26. If X is an $\alpha-connected$ space and $f:X\to Y$ is an $\alpha-irresolute$ mapping with the $\alpha-closed$ graph, then f is constant.

Proof. Suppose that f is not constant. Then there exist two distinct points x_1,x_2 in X such that $f(x_1)\neq f(x_2)$. Since the graph $G(f)$ is $\alpha-closed$ and

$(x_1, f(x_2)) \notin G(f),$ there exist $\alpha-open$ sets U and V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap V = \phi.$ Since f is $\alpha-irresolute, U$ and $f^{-1}(V)$ are non-empty disjoint $\alpha-open$ sets. It follows that X is not $\alpha-connected$. Therefore f is constant.

IV.HAUSDORFF AND URYSOHN SPACES

Definition 27. A space X is called a Urysohn space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U, y \in V$ and $Cl(U) \cap Cl(V) = \phi.$

Theorem 28. Let $f:(X,\tau) \to (Y,\sigma)$ be an $\alpha-weakly$ continuous injection and (Y,σ^{α}) be a Urysohn space. Then (X,τ^{α}) is a $T_2-space.$

Proof. For any distinct points $x_1, x_2 \in X,$ we have $f(x_1) \neq f(x_2)$ because f is injection. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1,$ $f(x_2) \in V_2$ and $Cl(V_1) \cap Cl(V_2) = \phi.$ We know that $\tau \subseteq \tau^{\alpha}$ which implies that $Cl_{\alpha}(V_i) \subseteq Cl(V_i)$ for $i = 1, 2.$ It follows that $Cl_{\alpha}(V_1) \cap Cl_{\alpha}(V_2) = \phi.$ Hence we have $Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_1))\right] \cap Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_2))\right] = \phi.$ Since f is $\alpha-weakly$ continuous, so by Theorem 8, we have $x_j \in f^{-1}(V_j) \subseteq Int_{\alpha}\left[f^{-1}(Cl_{\alpha}(V_j))\right]$ for $j = 1, 2.$ This implies that (X,τ^{α}) is $T_2.$

Theorem 29. Let $f:(X,\tau) \to (Y,\sigma)$ be an $\alpha-weakly$ continuous injection mapping and (Y,σ) be a $T_2-space.$ Then the graph $G(f)$ is an $\alpha-closed$ set of $X \times Y.$

Proof. Let $(x, y) \notin G(f).$ Then, we have $y \neq f(x).$ Since (Y,σ) is $T_2,$ there exist disjoint open sets S and T such that $f(x) \in S$ and $y \in T.$ Since f is $\alpha-weakly$ continuous, there exists an $\alpha-open$ set R containing x such that $f(R) \subseteq Cl_{\alpha}(S).$ Since S and T are disjoint, we have $T \cap Cl_{\alpha}(S) = \phi$ and hence $T \cap f(R) = \phi$. This shows that $(R \times T) \cap G(f) = \phi.$ It follows that $G(f)$ is $\alpha-closed$.

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