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On Lacunary *p*-Absolutely Summable Fuzzy Real-valued Triple Sequence Space

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Abstract— **In this article, we introduce the class of p-absolutely summable fuzzy real-valued triple sequence space $\left({}_3\ell_p\right)^F_\theta$. We make an effort to study some basic algebraic and topological properties of the introduced sequence space, namelysolid, monotone, symmetric, convergence free, sequence algebra are studied. Further, we investigate some relation with the class of p-Cesàro summable triple sequences and some other important inclusion results.**

Index terms - Fuzzy real numbers, lacunary sequence, triple sequences, symmetric, convergence free, sequence algebra.

I. INTRODUCTION

The fuzzy set theory extended the basic mathematical concept of a set. After the pioneering work done on fuzzy set theory by L. A. Zadeh [32] in 1965, a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modeling, uncertainty and vagueness in various problems arising in the field of science and engineering. Several mathematicians have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. In fact the fuzzy set theory has become an active area of research in science and engineering for the last 51 years. While studying fuzzy topological spaces, we face many situations where we need to deal with convergence of fuzzy numbers.

Agnew [1] studied the summability theory of multiple sequences and proved certain theorems for double sequences. At the initial stage, the different types of notions of triple sequences were introduced and investigated by Sahiner *et al.* [20] and Sahiner and Tripathy [21]. Savas and Esi [25] have introduced statistical convergence of triple sequences on probabilistic normed space. Esi [9] introduced statistical convergence of triple sequences in topological groups. Recently more works on triple sequences are done by Kumar *et al.* [14], Dutta *et al.* [6], Tripathy and Goswami [31], Nath and Roy [15-18] and many others.

Fridy and Orhan [11] introduced the concept of Lacunary statistical convergence. In the recent past, different classes of Lacunary sequences have been studied by some renowned researchers namely, K. Demirci [5], Bligin [4], Altin *et al*. [2], Altin [3], Gokhan *et al*. [12], Subramanian and Esi [26], Esi [8], Savas [24], Tripathy and Baruah [27], Dutta *et al*. [7], Tripathy and Dutta [28] etc.

A fuzzy number on R is a function $X : R \to L(=[0,1])$ associating each real number $t \in R$ having grade of membership X (t). Every real number r can be expressed as a fuzzy number $\bar{r}$ as:

$$\bar{r}(t) = \begin{cases} 1, & if \quad t = r \\ 0, & otherwise \end{cases}$$

The α-level set of a fuzzy number X, $0 < \alpha \le 1$, is defined and denoted as $[X]^\alpha = \{t \in R : X(t) \ge \alpha\}$.

A fuzzy number X is said to be convex if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

A fuzzy number X is called normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. If for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R, then a fuzzy number X is called upper semi-continuous. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$, whose additive and multiplicative identities are $\bar{0}$ and $\bar{1}$ respectively.

If D denotes the set of all closed bounded intervals $X = \left[X^L, X^R\right]$ on the real line R and if $d(X,Y) = \max(|X^L - X^R|, |Y^L - Y^R|)$, then (D, d) is a complete metric space. Also $\bar{d} : R(L) \times R(L) \to R$ defined by $\bar{d}(X,Y) = \sup_{0\le\alpha\le1} d([X]^\alpha, [Y]^\alpha)$, for $X, Y \in R(L)$ is also a metric on $R(L)$.

A lacunary sequence is an increasing integer sequence $\theta = \langle k_r \rangle$ $(r = 0,1,2,3,......)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be defined by $J_r = (k_{r-1}, k_r]$ and the ratio $\dfrac{k_r}{k_{r-1}}$ will be defined by q_r.

A lacunary sequence $\theta' = k'(r)$ is said to be lacunary refinement of the lacunary sequence $\theta = \langle k_r \rangle$ if $k_r \subset k'(r)$.

II. PRELIMINARIES AND BACKGROUND

In this section, some fundamental notions and definitions are defined, which are closely related to the paper. Throughout *N*, *R* and C denote the sets of natural and real numbers respectively.

Definition 2.1- A triple sequence is a function $x : N \times N \times N \to R(C)$. A fuzzy real-valued triple sequence $X = \langle X_{mnl} \rangle$ is a triple infinite array of fuzzy numbers $X_{mnl} \in R(L)$ for all $m, n, l \in N$. We denote the class of all fuzzy real-valued triple sequences by ${}_3(w^F)$.

Definition 2.2- A fuzzy real-valued triple sequence $X = \langle X_{mnl} \rangle$ is said to be convergent in Pringsheim's sense to the fuzzy number X, if for every $\varepsilon > 0$, $\exists$ $m_0 = m_0(\varepsilon), n_0 = n_0(\varepsilon), l_0 = l_0(\varepsilon) \in N$ such that $\bar{d}(X_{mnl}, X) < \varepsilon$, for all $m \ge m_0, n \ge n_0, l \ge l_0$.

Definition 2.3- A triple sequence $\theta_{r,s,t} = \{(m_r, n_s, l_t)\}$ $(r, s, t = 0, 1, 2, ...)$ of positive integers is said to be lacunary if there exists three increasing sequences of integers $\{m_r\}, \{n_s\}, \{l_t\}$ such that

$$m_0 = 0, h_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty$$

$$n_0 = 0, h_r = n_r - n_{r-1} \to \infty \text{ as } r \to \infty$$

$$l_0 = 0, h_r = l_r - l_{r-1} \to \infty \text{ as } r \to \infty.$$

Let us denote $m_{r,s,t} = m_r n_s l_t$ and $h_{r,s,t} = h_r h_s h_t$ and the intervals are determined by $\theta_{r,s,t}$ and it will be defined by

$$J_{r,s,t} = \{(m,n,l) : m_{r-1} < m \le m_r, \ n_{s-1} < n \le n_s, \ l_{t-1} < l \le l_t\}$$

and $q_r = \frac{m_r}{m_{r-1}}$, $q_s = \frac{n_s}{n_{s-1}}$, $q_t = \frac{l_t}{l_{t-1}}$.

Definition 2.4- A triple sequence $\langle x_{mnl} \rangle$ is said to be $\theta_{r,s,p}$ convergent to *L* if for every $\varepsilon > 0$ and there exists integers $n_0 \in N$ such that $\frac{1}{h_{r,s,t}} \sum_{(m,n,l) \in J_{r,s,t}} \bar{d}(x_{mnl}, L) < \varepsilon \ \forall r, s, t \ge n_o$

$$\therefore \theta_{r,s,t} - \lim x_{mnl} = L.$$

Definition 2.5- A fuzzy real-valued triple sequence $X = \langle X_{mnl} \rangle$ is said to be convergent in Pringsheims sense to the fuzzy real number *X*, if for every $\varepsilon > 0$, $\exists$ $m_0 = m_0(\varepsilon), n_0 = n_0(\varepsilon)$, $l_0 = l_0(\varepsilon) \in N$ such that $\bar{d}(X_{mnl}, X) < \varepsilon$ for all $m \ge m_0, n \ge n_0, l \ge l_0$.

Definition 2.6- A fuzzy real-valued triple sequence $X = \langle X_{mnl} \rangle$ is said to be bounded, if $\sup_{m,n,l} \bar{d}(X_{mnl}, \bar{0}) < \infty$.

Definition 2.7- A fuzzy real-valued triple sequence space ${}_3(w^F)$ is said to be solid if $\langle Y_{mnl} \rangle \in_3 (w^F)$ whenever $\langle X_{mnl} \rangle \in_3 (w^F)$ and $\bar{d}(Y_{mnl}, \bar{0}) \le \bar{d}(X_{mnl}, \bar{0})$ for all $m, n, l \in N$.

Definition 2.8- A fuzzy real-valued triple sequence space ${}_3(w^F)$ is said to be monotone if it contains the canonical pre-image of all its step spaces.

Definition 2.9- A fuzzy real-valued triple sequence space ${}_3(w^F)$ is said to be symmetric if $\langle X_{\pi(mnl)} \rangle \in_3 (w^F)$, whenever $\langle X_{mnl} \rangle \in_3 (w^F)$ where π is a permutation on $N \times N \times N$.

Definition 2.10- A fuzzy real-valued triple sequence space ${}_3(w^F)$ is said to be convergence free if $\langle Y_{mnl} \rangle \in_3 (w^F)$ whenever $\langle X_{mnl} \rangle \in_3 (w^F)$ and $X_{mnl} = \bar{0}$ implies $Y_{mnl} = \bar{0}$.

Definition 2.11- A fuzzy real-valued triple sequence space ${}_3(w^F)$ is said to sequence algebra if $\langle X_{mnl} \otimes Y_{mnl} \rangle \in_3 (w^F)$, whenever $\langle X_{mnl} \rangle, \langle Y_{mnl} \rangle \in_3 (w^F)$.

Definition 2.12- A fuzzy real-valued triple sequence $\langle X_{mnl} \rangle$ is said to be Cesáro summable to a fuzzy real number *L*, if $\bar{d}\left(\frac{1}{uvw} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{l=1}^{w} X_{mnl}, L \right) \to 0$ as $u, v, w \to \infty$.

Definition 2.13- A fuzzy real-valued triple sequence $\langle X_{mnl} \rangle$ is said to be strongly p-Cesáro summable to a fuzzy real number *L* if $\frac{1}{uvw} \left(\sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{l=1}^{w} \left[\bar{d}(X_{mnl}, L)\right]^p \right) \to 0$ as $u, v, w \to \infty$. and we denote it by $Ces_3(p)$.

The class of fuzzy real-valued triple sequences ${}_3\ell^F(p)$ introduced by Nath and Roy [14] as follows:

$${}_3\ell^F(p) = \left\{ X = (X_{nkl}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left[\bar{d}(X_{mnl}, \bar{0})\right]^{p_{nkl}} < \infty \right\}.$$

where $p = \langle p_{mnl} \rangle$ is a triple sequence of bounded strictly positive numbers.

We introduce the class of fuzzy real-valued triple sequences $\left({}_3\ell_p\right)_\theta^F$ as follows:

$$\left({}_3\ell_p\right)_\theta^F = \left\{ X = (X_{mnl}) : \sum_{r,s,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty \right\},$$

$1 \le p < \infty$.

To prove some results in the paper, the following existing result will be used.

Lemma 2.1- Every normal sequence space is monotone.

III. MAIN RESULTS

In this section, we examine some basic topological and algebraic properties of the introduced sequence space and obtain some inclusion relation related to the space.

Theorem 3.1- The class of sequence space $\left({}_3\ell_p\right)_\theta^F$ *is closed under addition and scalar multiplication operations.*

Proof. Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ be a triple lacunary sequence and $\langle X_{mnl} \rangle \in \left({}_3\ell_p\right)_\theta^F$.

Since $\bar{d}\left(cX_{mnl}^\alpha, cY_{mnl}^\alpha \right) = |c| \bar{d}\left(X_{mnl}^\alpha, Y_{mnl}^\alpha \right)$, we have $\bar{d}\left(cX, cY \right) = |c|\bar{d}\left(X, Y\right)$, for any $c \in R$.

This gives

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(cX_{mnl}, \bar{0}\right) \right)^p$$

$$= \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} |c|\, \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p$$

$$= |c|^p \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty.$$

$\therefore \langle cX_{mnl} \rangle \in \left({}_3\ell_p\right)_\theta^F$.

Next let

$\langle X_{mnl} \rangle, \langle Y_{mnl} \rangle \in \left({}_3\ell_p\right)_\theta^F$.

$$\because \bar{d}\left(X_{mnl}^\alpha + Y_{mnl}^\alpha, X_0^\alpha + Y_0^\alpha\right) \le \bar{d}\left(X_{mnl}^\alpha + X_0^\alpha\right) + \bar{d}\left(Y_{mnl}^\alpha + Y_0^\alpha\right),$$

$$\therefore \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left\{(X_{mnl} + Y_{mnl}), \bar{0}\right\} \right)^p \le$$

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p + \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(Y_{mnl}, \bar{0}\right) \right)^p < \infty.$$

Thus $\langle X_{mnl} + Y_{mnl} \rangle \in \left({}_3\ell_p\right)_\theta^F$. ■

Theorem 3.2- Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ *be a triple lacunary sequence and* $\liminf q_r > 1$, $\liminf q_s' > 1$, *and* $\liminf q_t'' > 1$, *then for* $0 < p < 1$, $Ces_3(p) \subset \left({}_3\ell_p\right)_\theta^F$.

Proof. Let $\liminf q_r > 1$, $\liminf q_s' > 1$ and $\liminf q_t'' > 1$.

Then there exists $\delta > 0$ such that $q_r > 1 + \delta$, $q_s' > 1 + \delta$ and $q_t'' > 1 + \delta$.

This implies $\frac{m_r}{h_r} = \frac{m_r}{m_r - m_{r-1}} = \frac{q_r}{q_r - 1} \le \frac{\delta}{1+\delta}$, $\frac{n_s}{h_s'} = \frac{n_s}{n_s - n_{s-1}} = \frac{q_s'}{q_s' - 1} \le \frac{\delta}{1+\delta}$, $\frac{l_t}{h_t''} = \frac{l_t}{l_t - l_{t-1}} = \frac{q_t''}{q_t'' - 1}, \le \frac{\delta}{1+\delta}$.

Let $\langle X_{mnl} \rangle \in Ces_3(p)$. Then we can write

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \frac{1}{m_r n_s l_t} \left\{ \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) \right\}^p < \infty. \quad \text{.........................} \quad (1)$$

We have

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p$$

$$= \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left[\frac{1}{h_{r,s,t}} \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) - \frac{1}{h_{r,s,t}} \sum_{m=1,n-1,l=1}^{m_{r-1},n_{s-1},l_{t-1}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p$$

$$\le \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left[\frac{1}{h_{r,s,t}} \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p -$$

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left[\frac{1}{h_{r,s,t}} \sum_{m=1,n-1,l=1}^{m_{r-1},n_{s-1},l_{t-1}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p$$

$$= \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left[\frac{1}{h_r h_s' h_t''} \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p - \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left[\frac{1}{h_r h_s' h_t''} \sum_{m=1,n-1,l=1}^{m_{r-1},n_{s-1},l_{t-1}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p$$

$$= \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \frac{1}{m_r n_s l_t} \left[\frac{m_r}{h_r} \frac{n_s}{h_s'} \frac{l_t}{h_t''} \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p -$$

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \frac{1}{m_{r-1} n_{s-1} l_{t-1}} \left[\frac{m_{r-1}}{h_r} \frac{n_{s-1}}{h_s'} \frac{l_{t-1}}{h_t''} \sum_{m=1,n-1,l=1}^{m_{r-1},n_{s-1},l_{t-1}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p$$

$$\le \left(\frac{1+\delta}{\delta}\right)^2 \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \frac{1}{m_r n_s l_t} \left[\sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p -$$

$$\left(\frac{1}{\delta}\right)^2 \sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \frac{1}{m_{r-1} n_{s-1} l_{t-1}} \left[\sum_{m=1,n-1,l=1}^{m_{r-1},n_{s-1},l_{t-1}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right]^p$$

From (1), we have

$$\sum_{r,s,t=1,1,1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty.$$

$\therefore \langle X_{mnl} \rangle \in \left({}_3\ell_p\right)_\theta^F$.

This implies $Ces_3(p) \subset \left({}_3\ell_p\right)_\theta^F$. ■

Theorem 3.3- Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ *be a triple lacunary sequence and lim sup* $q_r < \infty$, *lim sup* $q_s' < \infty$, *and lim sup* $q_t'' < \infty$, *then for* $0 < p < 1$, $\left({}_3\ell_p\right)_\theta^F \subset Ces_3(p)$.

Proof. Let *lim sup* $q_r < \infty$, *lim sup* $q'_s < \infty$ and *lim sup* $q''_t < \infty$.

Then there exists $M > 0$ such that $q_r < M, q'_s < M$ and $q''_t < M$ for all *r, s, t*.

Let $\langle X_{mnl} \rangle \in ({}_3\ell_p)^F_\theta$ and $\varepsilon > 0$ be given, then there exist $r_0 > 0, s_0 > 0$ and $t_0 > 0$ such that .

Let $A_{ijk} = \frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} [\bar{d}(X_{mnl}, \bar{0})]^p < \varepsilon$, for every $i > r_0$, $j > s_0$ and $k > t_0$.

Let $K = \max\{A_{ijk} : 1 \le r \le r_0; 1 \le s \le s_0; 1 \le t \le t_0\}$ and choose *a*, *b* and *c* such that $m_{r-1} \le a \le m_r$, $n_{s-1} \le b \le n_s$ and $l_{t-1} \le c \le l_t$.

Then we have

$$\frac{1}{abc} \sum_{m=1,n-1,l=1}^{a,b,c} [\bar{d}(X_{mnl}, \bar{0})]^p \le \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{m=1,n-1,l=1}^{m_r,n_s,l_t} [\bar{d}(X_{mnl}, \bar{0})]^p$$

$$\le \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{u=1,v=1,w=1}^{r,s,t} \left\{ \sum_{m,n,l \in J_{u,v,w}} [\bar{d}(X_{mnl}, \bar{0})]^p \right\}$$

$$= \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{u=1,v=1,w=1}^{r_0,s_0,t_0} h_{u,v,w} A_{u,v,w} + \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)\cup(t_0<w<t)} h_{u,v,w} A_{u,v,w}$$

$$\le \frac{K}{m_{r-1}n_{s-1}l_{t-1}} \sum_{u=1,v=1,w=1}^{r_0,s_0,t_0} h_{u,v,w} + \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)\cup(t_0<w<t)} h_{u,v,w} A_{u,v,w}$$

$$\le \frac{K m_{r_0} n_{s_0} l_{t_0} r_0 s_0 t_0}{m_{r-1}n_{s-1}l_{t-1}} + \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)\cup(t_0<w<t)} h_{u,v,w} A_{u,v,w}$$

$$\le \frac{K m_{r_0} n_{s_0} l_{t_0} r_0 s_0 t_0}{m_{r-1}n_{s-1}l_{t-1}} + \sup_{(u>r_0)\cup(v>s_0)\cup(w>t_0)} (A_{u,v,w}) . \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \sum_{(r_0<u<r)\cup(s_0<v<s)\cup(t_0<w<t)} h_{u,v,w}$$

$$\le \frac{K m_{r_0} n_{s_0} l_{t_0} r_0 s_0 t_0}{m_{r-1}n_{s-1}l_{t-1}} + \frac{1}{m_{r-1}n_{s-1}l_{t-1}} \varepsilon \sum_{(r_0<u<r)\cup(s_0<v<s)\cup(t_0<w<t)} h_{u,v,w}$$

$$\le \frac{K m_{r_0} n_{s_0} l_{t_0} r_0 s_0 t_0}{m_{r-1}n_{s-1}l_{t-1}} + \varepsilon M^3.$$

We note that m_r, n_s and l_t will tend to infinity as $a, b, c \to \infty$, thereby

$$\frac{1}{abc} \sum_{m=1,n-1,l=1}^{a,b,c} [\bar{d}(X_{mnl}, \bar{0})]^p \to \infty.$$

$$\therefore \langle X_{mnl} \rangle \in Ces_3(p).$$

This implies $({}_3\ell_p)^F_\theta \subset Ces_3(p)$. ■

Theorem 3.4- Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ *be a triple lacunary sequence. If* $1 < \liminf q_r < \limsup q_r < \infty$, $1 < \liminf q'_s < \limsup q'_s < \infty$ *and* $1 < \liminf q''_t < \limsup q''_t < \infty$, *then* $({}_3\ell_p)^F_\theta = Ces_3(p)$.

Proof. From the *Theorem 3.2 and Theorem 3.3*, the result easily follows. ■

Theorem 3.5- The class of sequences $({}_3\ell_p)^F_\theta$ *is normal and monotone.*

Proof. Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ be a triple lacunary sequence. Let consider the triple sequences $\langle X_{mnl} \rangle, \langle Y_{mnl} \rangle \in_3 (w^F)$ such that $\bar{d}(Y_{mnl}, \bar{0}) \le \bar{d}(X_{mnl}, \bar{0})$ for all $m, n, l \in N$.

Let $\langle X_{mnl} \rangle \in ({}_3\ell_p)^F_\theta$, then $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty.$

We have

$$\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(Y_{mnl}, \bar{0}\right) \right)^p \le \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty.$$

$$\therefore \langle Y_{mnl} \rangle \in ({}_3\ell_p)^F_\theta.$$

Hence the class of sequences $({}_3\ell_p)^F_\theta$ is normal. Now by *Lemma 2.1*, the space is monotone. ■

Theorem 3.6- The class of sequences $({}_3\ell_p)^F_\theta$ *is sequence algebra.*

Proof. Let $\theta_{r,s,t} = \{ m_r, n_s, l_t \}$ be a triple lacunary sequence and $\langle X_{mnl} \rangle, \langle Y_{mnl} \rangle \in ({}_3\ell_p)^F_\theta$.

Then $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl}, \bar{0}\right) \right)^p < \infty,$

and $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(Y_{mnl}, \bar{0}\right) \right)^p < \infty.$

Now $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}\left(X_{mnl} \otimes Y_{mnl}, \bar{0}\right) \right)^p.$

$$\le \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty} \left(\frac{1}{h_{r,s,t}} \sum_{m,n,l \in J_{r,s,t}} \bar{d}(X_{mnl}, \bar{0}) \bar{d}(Y_{mnl}, \bar{0}) \right)^p.$$

$$\leq \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}(X_{mnl},\bar{0})\right)^p \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}(Y_{mnl},\bar{0})\right)^p < \infty.$$

Hence the class of sequences $\left({}_3\ell_p\right)_\theta^F$ is sequence algebra. ■

Proposition 3.7 - The class of sequences $\left({}_3\ell_p\right)_\theta^F$ *is not convergence free in general.*

Proof. The proof of result follows from the following example.

Let $\theta_{r,s,t} = \left(3^r, 3^s, 3^p\right)$ be a triple lacunary sequence.

Consider the double sequence $\langle X_{mnl}\rangle$ defined by

$$X_{mnl}(t) = \begin{cases} \{1+(m+n+l)^2 t\}, & \text{for } -\frac{1}{(m+n+l)^2} \leq t \leq 0; \\ \{1-(m+n+l)^2 t\}, & \text{for } 0 < t \leq \frac{1}{(m+n+l)^2}; \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}\left(X_{mnl},\bar{0}\right)\right)^p = \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{4.3^{r+s+t-2}}\sum_{m,n,l\in J_{r,s,t}}(m+n+l)^{-2}\right) < \infty.$$

$\therefore \langle X_{mnl}\rangle \in \left({}_3\ell_p\right)_\theta^F$.

Now consider the double sequence $\langle Y_{mnl}\rangle$ defined by

$$Y_{mnl}(t) = \begin{cases} \{1+t\sqrt{m+n+l}\}, & \text{for } -\frac{1}{\sqrt{m+n+l}} \leq t \leq 0; \\ \{1-t\sqrt{m+n+l}\}, & \text{for } 0 < t \leq \frac{1}{\sqrt{m+n+l}}; \\ 0, & \text{otherwise} \end{cases}$$

Then $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}\left(Y_{mnl},\bar{0}\right)\right)^p$

$$= \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{4.3^{r+s+t-2}}\sum_{m,n,l\in J_{r,s,t}}(m+n+l)^{-\frac{1}{2}}\right) = \infty.$$

$\therefore \langle Y_{mnl}\rangle \notin \left({}_3\ell_p\right)_\theta^F$.

Hence the sequence space $\left({}_3\ell_p\right)_\theta^F$ is not convergent free. ■

Theorem 3.8- ${}_3\ell^F(p) \subset \left({}_3\ell_p\right)_\theta^F$, if $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\frac{1}{h_{r,s,t}} < \infty$.

Proof. Let $\langle X_{mnl}\rangle \in {}_3\ell^F(p)$.

We choose $m_0 > 0$ such that $\sum_{m,n,l>m_0}\left[\bar{d}(X_{mnl},\bar{0})\right]^p < 1,$ for all $m,n,l > m_0$.

This implies $\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}\left(X_{mnl},\bar{0}\right)\right)^p < \infty.$

$\therefore \langle X_{mnl}\rangle \in \left({}_3\ell_p\right)_\theta^F$.

Hence ${}_3\ell^F(p) \subset \left({}_3\ell_p\right)_\theta^F$. ■

Theorem 3.9- For $0 < p < q$, $\left({}_3\ell_p\right)_\theta^F \subset \left({}_3\ell_q\right)_\theta^F$.

Proof. From the following inclusion relation:

$$\sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}\left(X_{mnl},\bar{0}\right)\right)^p \subset \sum_{r=1,s=1,t=1}^{\infty,\infty,\infty}\left(\frac{1}{h_{r,s,t}}\sum_{m,n,l\in J_{r,s,t}}\bar{d}\left(X_{mnl},\bar{0}\right)\right)^q.$$

the result follows. ■

IV. CONCLUSION

For the development of any sequence space, convergence of that sequence space plays an important role. In this research work, we have introduced and studied the notion of the class of *p*-absolutely summable fuzzy real-valued triple sequence space $\left({}_3\ell_p\right)_\theta^F$. Some fundamental algebraic and topological properties of this sequence space are established. The introduced notion can be applied for further investigations from different aspects.

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